ON MULTIPLE RECURRENCE AND OTHER PROPERTIES OF "NICE" INFINITE MEASURE PRESERVING TRANSFORMATIONS

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Abstract. We discuss multiple versions of rational ergodicity and rational weak mixing for "nice" transformations, including Markov shifts, certain interval maps and hyperbolic geodesic flows. These properties entail multiple recurrence.

§1. Introduction: multiple properties

The measure preserving transformation (`MPT`) $(X, \mathcal{B}, m, T)$ is called

- d-`recurrent` as in [7] if $\forall\ A \in \mathcal{B},\ m(A) > 0\ \exists\ n \geq 1$ so that

$$m(A \cap T^{-n}A \cap T^{-2n}A \cap \cdots \cap T^{-dn}A) > 0;$$

1-recurrence being equivalent to conservativity.

In [7], the authors considered the multiple recurrence of the **Markov shift** $(X, \mathcal{B}, m, T)$ of the stochastic matrix $P \colon S \times S \to [0, 1]$ with invariant distribution $\{\mu_s \colon s \in S\}$ where

$X = S^{\mathbb{Z}},\ T = \mathtt{shift}, \mathcal{B} = \sigma(\{\mathtt{cylinders}\})$ &

$$m([s_0, \ldots, s_n]_k) = \mu_{s_0} p_{s_0,s_1} \ldots p_{s_{n-1},s_n}$$

where the *cylinder* $[s_0, \ldots, s_n]_k := \{x \in X \colon x_{k+j} = s_j\ \forall 0 \leq j \leq n\}$,

showing for $d \in \mathbb{N}$ that if $(X, \mathcal{B}, m, T)$ is conservative, ergodic, then

- T is d-recurrent $\Leftrightarrow \underbrace{T \times \ldots \times T}_{d\text{-times}}$ is conservative, ergodic.

For Markov shifts, d-recurrence is equivalent to d-`rational ergodicity`:

The conservative, ergodic, measure preserving transformation (`CEMPT`) $(X, \mathcal{B}, m, T)$ is called *d-rationally ergodic along* $\mathfrak{K} \subset \mathbb{N}$ if there exist:

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(i) a sequence of constants $a_d(n) \uparrow\ \infty$ and

(ii) a dense, T-invariant, hereditary ring (aka ideal) $R_d(T) \subset \mathcal{F} := \{F \in \mathcal{B} :\ m(F) < \infty\}$ s.t.

$$\frac{1}{a_d(n)} \sum_{k=0}^{n-1} m(\bigcap_{j=0}^{d} T^{-(jk+r_j)} B_j) \xrightarrow[n\to\infty,\ n\in\mathfrak{K}]{} \prod_{j=0}^{d} m(B_j)$$
$$\forall\ B_0, B_1, \ldots, B_d \in R_d(T)\ \&\ r_0, \ldots, r_d \in \mathbb{Z};$$

For Markov shifts,

$$u_d(n) = u_n^d, \quad a_d(n) = \sum_{k=1}^{n} u_n^d$$

where $u_n := \frac{p_{s,s}^{(n)}}{\mu_s}$ (any $s \in S$) and

$$R_d(T) \supset \{B \in \mathcal{F} :\ B \subset C \texttt{ a cylinder}\}.$$

In this paper we extend this to further classes of transformations which we call "`nice`".

Some notes on terminology:

- *d-rational ergodicity* is d-rational ergodicity along $\mathbb{N}$;
- *subsequence d-rational ergodicity* is d-rational ergodicity along some $\mathfrak{K} \subset \mathbb{N}$;
- 1-rational ergodicity is called *weak rational ergodicity* in [1].

Evidently, subsequence d-rational ergodicity implies d-recurrence

Description of results.

In §2, we define "d-nice transformations" and show that these are subsequence d-rational ergodic. This is applied in §5 where we establish 2-recurrence [`2-dissipation`] of the geodesic flow of a $\mathbb{Z}$-cover [$\mathbb{Z}^2$`-cover`] of a compact, hyperbolic surface, as advertised in [18]. In §3, we give sufficient conditions for the stronger multiple rational weak mixing properties which are applied to show 1-rational weak mixing

- of certain special semiflows in §4;

- of the geodesic flow of a $\mathbb{Z}^\kappa$-cover ($\kappa = 1, 2$) of a compact, hyperbolic surface.

Asymptotic equivalence of sequences. Throughout the paper we use the following notations for sequences $(a_1, a_2, \dots),\ (b_1, b_2, \dots) \in \mathbb{R}^{\mathbb{N}}$:

- $a_n \approx b_n$ if $a_n - b_n \xrightarrow[n\to\infty]{} 0$,
- for $M > 0$, $a_n = b_n \pm M$ if $|a_n - b_n| \le M \ \forall\ n \ge 1$;
  and for $(a_1, a_2, \dots),\ (b_1, b_2, \dots) \in \mathbb{R}_+^{\mathbb{N}}$:
- $a_n \sim b_n$ if $\frac{a_n}{b_n} \xrightarrow[n\to\infty]{} 1$,
- $a_n \propto b_n$ if $\frac{a_n}{b_n} \xrightarrow[n\to\infty]{} c \in \mathbb{R}_+$,
- $a_n \ll b_n$ if $\exists\ M > 0$ so that $a_n \le M b_n \ \forall\ n \ge 1$,
- $a_n \asymp b_n$ if $a_n \ll b_n$ and $b_n \ll a_n$.
  For $M > 1$ we also write
- $a_n = M^{\pm 1} b_n$ or $a_n \overset{M}{\asymp} b_n$ if $\frac{b_n}{M} \le a_n \le M b_n \ \forall\ n \ge 1$.

§2 Nice transformations

Let $(X, \mathcal{B}, m, T)$ be a `CEMPT`.

We'll call a set $\Omega \in \mathcal{F}_+ := \{A \in \mathcal{B} :\ 0 < m(A) < \infty\}$ *admissible* for T if
$m(\bigcap_{k=0}^{d} T^{-kn}\Omega) \asymp u(n)^d \ \ \forall\ d \ge 1$ where $u(n) = u(\Omega, n) := \frac{m(\Omega\cap T^{-n}\Omega)}{m(\Omega)}$.

Let $d \in \mathbb{N}$. We'll call the `CEMPT` $(X, \mathcal{B}, m, T)$ *d-nice* if

(i) there is an admissible set $\Omega \in \mathcal{F}_+$ for T;

(ii) $(X_d, \mathcal{B}_d, m_d, T_d) := (X^d, \mathcal{B}(X^d), \underbrace{m \times \dots \times m}_{d\text{-times}}, \underbrace{T \times \dots \times T}_{d\text{-times}})$ is a `CEMPT`;

(iii) $\exists\ M > 1$ and a countable, dense collection $\mathcal{A} \subset \mathcal{B} \cap \Omega$ with $\Omega \in \mathcal{A}$ s.t. $\forall\ r_0, \dots, r_d \in \mathbb{Z}$,

$$\sum_{k=0}^{n-1} m(\bigcap_{k=0}^{d} T^{-kn+r_k} B_k) \overset{M}{\asymp} \prod_{k=0}^{d} m(B_k) a_d(n) \ \forall\ B_0, B_1, \dots, B_d \in \mathcal{A}$$

where $a_d(n) := \sum_{k=1}^{n} u(\Omega, k)^d \to \infty$ by condition (ii),

(iv) $a_d(2n) \ll a_d(n)$.

Proposition 2.1 *If is T d-nice, then $a_d(n) \to \infty$ and T_d is 1-rationally ergodic with return sequence $a_n(T_d) \asymp a_d(n)$.*

Proof For $\Omega \in \mathcal{F}_+$ admissible,

$$\int_{\Omega^d} S_n^{(T_d)}(1_{\Omega^d})^2 dm_d \ll \left(\sum_{k=1}^n u(\Omega,k)^d\right)^2 \asymp \left(\int_{\Omega^d} S_n^{(T_d)}(1_{\Omega^d}) dm_d\right)^2. \quad ☑$$

Theorem 2.2

If $(X,\mathcal{B},m,T)$ is d-nice, then it is subsequence d-rationally ergodic. and the hereditary ring satisfies

$$R_d(T) \supset \{B \in \mathcal{F} :\ \exists\, n \in \mathbb{Z},\ B \subset T^n\Omega\}.$$

The proof of theorem 2.2 uses:

Lemma 2.3 *Let $(X,\mathcal{B},m,T)$ be d-nice and let $\Omega \in \mathcal{F}_+$ be admissible. For any $0 \le \nu \le d$, define*

$$\psi_n^{(\nu)} := \sum_{k=1}^n \prod_{i=1}^{\nu} 1_\Omega \circ T^{-ik} \cdot \prod_{j=1}^{d-\nu} 1_\Omega \circ T^{jk},$$

then

$$\text{(i)} \quad \int_\Omega \psi_n^{(\nu)} dm \asymp\ a_d(n) \ \ \& \ \ \text{(ii)} \quad \int_\Omega (\psi^{(\nu)})^2 dm = O\Big(a_d(n)^2\Big).$$

Proof of lemma 2.3. This lemma is a generalization of lemma 1.5 in [7].

Throughout, we use admissibility of Ω:
if $b(1),\dots,b(\kappa) \in \mathbb{Z}$ and $b(1) \le b(2) \le \dots \le b(\kappa)$ then

$$(0)\ \ m\Big(\bigcap_{r=1}^{\kappa} T^{-b(r)}\Omega\Big) = m\Big(\bigcap_{r=1}^{\kappa} T^{b(r)}\Omega\Big) \asymp \prod_{r=2}^{\kappa} m(\Omega \cap T^{-(b(r)-b(r-1))}\Omega).$$

Set

$$\epsilon_k(\nu) := \prod_{-\nu \le j \le d-\nu,\ j \ne 0} 1_\Omega \circ T^{jk},$$

then

$$\psi_n^{(\nu)} = \sum_{k=1}^n \epsilon_k(\nu) \text{ and } \quad \int_\Omega (\psi_n^{(\nu)})^2 dm \le 2\sum_{k=1}^n \sum_{\ell=k}^n \int_\Omega \epsilon_k(\nu)\epsilon_\ell(\nu) dm.$$

In view of (0), the form of $\int_\Omega \epsilon_k(\nu)\epsilon_\ell(\nu)dm$ depends on the orders of the sets $\{ik, j\ell:\ 1 \le i,j \le \nu\}$ and $\{ik, j\ell:\ 1 \le i,j \le d-\nu\}$.

The ordering of

$$\Omega_d(k,\ell) := \{ik,\ j\ell:\ 1 \le i,j \le d\} \quad (d,k,\ell \in \mathbb{N})$$

is treated in lemma 1.5 of [7] and we'll use results established there.

Define $N_{(k,\ell)}: \mathbb{N} \times \{0,1\} \to \mathbb{N}$ by $N_{k,\ell}(j,\epsilon) = (1-\epsilon)jk + \epsilon j\ell$, then for each $d \geq 1$, $N_{(k,\ell)}: \mathbb{N}_d \times \{0,1\} \to \Omega_d(k,\ell)$ is a surjection. Here $\mathbb{N}_d := \{1,2,\ldots,d\}$.

An *ordering* of $\Omega_d(k,\ell)$ is a bijection $\pi : \mathbb{N}_{2d} \to \mathbb{N}_d \times \{0,1\}$ so that $N_{(k,\ell)} \circ \pi : \mathbb{N}_{2d} \to \Omega_d(k,\ell)$ is non-decreasing.

Given a bijection $\pi : \mathbb{N}_{2d} \to \mathbb{N}_d \times \{0,1\}$, let

$$D(\pi) := \{(k,\ell) \in \mathbb{N} \times \mathbb{N}\ \pi \text{ orders } \Omega_d(k,\ell)\}.$$

Possibly $D(\pi) = \emptyset$. However, as shown in lemma 1.5 in [7], if

$$F_d := \{\frac{p}{q}:\ 0 \leq p \leq q \leq d,\ (p,q) = 1\} = \{0 := r_0^{(d)} < r_1^{(d)} < \cdots < r_{N_d}^{(d)} = 1\}$$

is the **Farey sequence** of order d, then for each $0 \leq j < N_d$, there is a bijection $\pi_j : \mathbb{N}_{2d} \to \mathbb{N}_d \times \{0,1\}$ so that

$$D(\pi_j) = \{(k,\ell) \in \mathbb{N}^2:\ \frac{k}{\ell} \in (r_j, r_{j+1}]\}. \tag{1}$$

Let $\mathfrak{o}_d := \{\pi_j :\ 0 \leq j < N_d\}$. It follows from (1) that if $1 \leq d' < d$ and $\pi \in \mathfrak{o}_d$, then $\exists\ \pi' \in \mathfrak{o}_{d'}$ such that $D(\pi) \subset D(\pi')$.

Fix $\pi \in \mathfrak{o}_d$, $(k,\ell) \in D(\pi)$, then writing $\pi(j) = (\kappa_j, \epsilon_j)$ for $1 \leq j \leq 2d$, we have (see [7])

$$\begin{aligned} N_{(k,\ell)} \circ \pi(j) - N_{(k,\ell)} \circ \pi(j-1) &= \kappa_j\big[(1-\epsilon_j)k + \epsilon_j\ell\big] - \kappa_{j-1}\big[(1-\epsilon_{j-1})k + \epsilon_{j-1}\ell\big] \\ &= \langle a_j, (k,\ell)\rangle \end{aligned}$$

where $N_{(k,\ell)} \circ \pi(0) := 0$, $a_1 = (\kappa_1(1-\epsilon_1), \kappa_1\epsilon_1)$ and

$$a_j = a_j(\pi) := (\kappa_j(1-\epsilon_j) - \kappa_{j-1}(1-\epsilon_{j-1}), \kappa_j\epsilon_j - \kappa_{j-1}\epsilon_{j-1}) \quad (j \geq 2).$$

The vectors $\{a_j(\pi)\}_{j=1}^{2d}$ are non-zero and

$$\{a_j(\pi):\ 1 \leq j \leq 2d\} = \{a_j^{(1)}(\pi), a_j^{(2)}(\pi):\ 1 \leq j \leq d\}$$

where $a_j^{(1)}(\pi)$ and $a_j^{(2)}(\pi)$ are linearly independent $\forall\ 1 \leq j \leq d\}$.

Now, set

$$\epsilon_k^{\pm}(\nu) = \prod_{j=1}^{\nu} 1_\Omega \circ T^{\pm jk},$$

then $\epsilon_k(\nu) = \epsilon_k^-(\nu)\epsilon_k^+(d-\nu)$,

$$\int_\Omega \epsilon_k(\nu)\epsilon_\ell(\nu)dm = \int_\Omega (e_k^-(\nu)\epsilon_\ell^-(\nu))(\epsilon_k^+(d-\nu)\epsilon_\ell^+(d-\nu))dm$$

and it follows from admissibility that

$$\int_\Omega (e_k^-(\nu)\epsilon_\ell^-(\nu))(\epsilon_k^+(d-\nu)\epsilon_\ell^+(d-\nu))dm \asymp \int_\Omega \epsilon_k^+(\nu)\epsilon_\ell^+(\nu)dm \int_\Omega \epsilon_k^+(d-\nu)\epsilon_\ell^+(d-\nu)dm.$$

Using the discussion above (and admissibility of Ω),

$$\int_\Omega \epsilon_k^+(d)\epsilon_\ell^+(d)dm \asymp \prod_{j=1}^{2d} u(\langle a_j,(k,\ell)\rangle)\ \forall\ (k,\ell)\in D(\pi),\ \pi\in\mathfrak{o}_d. \tag{2}$$

We now complete the proof of lemma 2.3 in case $\nu \geq d-\nu$ (the other case being similar).

For each $\pi\in\mathfrak{o}_\nu$, let $\pi'\in\mathfrak{o}_{d-\nu}$ be such that $D(\pi)\subset D(\pi')$.

Since $\{(k,\ell)\in\mathbb{N}^2\colon\ k\leq\ell\} = \biguplus_{\pi\in\mathfrak{o}_\nu} D(\pi)$, we have:

$$\begin{aligned}\int_\Omega (\psi_n^{(\nu)})^2 dm &\leq 2\sum_{k=1}^n\sum_{\ell=k}^n \int_\Omega \epsilon_k(\nu)\epsilon_\ell(\nu)dm\\ &= 2\sum_{\pi\in\mathfrak{o}_\nu}\ \sum_{(k,\ell)\in D(\pi),\ k,\ell\leq n}\int_\Omega \epsilon_k(\nu)\epsilon_\ell(\nu)dm.\end{aligned}$$

For each $\pi\in\mathfrak{o}_\nu$,

$$\begin{aligned}&\sum_{(k,\ell)\in D(\pi)\cap\mathbb{N}_n^2}\int_\Omega \epsilon_k(\nu)\epsilon_\ell(\nu)dm\\ &= \sum_{(k,\ell)\in D(\pi)\cap\mathbb{N}_n^2}\int_\Omega \epsilon_k^+(\nu)\epsilon_\ell^+(\nu)dm\int_\Omega \epsilon_k^+(d-\nu)\epsilon_\ell^+(d-\nu)dm\\ &\asymp \sum_{(k,\ell)\in D(\pi)\cap\mathbb{N}_n^2}\prod_{j=1}^{2\nu} u(\langle a_j(\pi),(k,\ell)\rangle)\prod_{j=1}^{2(d-\nu)} u(\langle a_j(\pi'),(k,\ell)\rangle)\\ &= \sum_{(k,\ell)\in D(\pi)\cap\mathbb{N}_n^2}\prod_{j=1}^{d} u(\langle a_j^{(1)},(k,\ell)\rangle)u(\langle a_j^{(2)},(k,\ell)\rangle)\end{aligned}$$

where

$$\{a_j^{(1)},a_j^{(2)}\}_{j=1}^{2d} = \{a_j^{(1)}(\pi),a_j^{(2)}(\pi)\}_{j=1}^{2\nu}\cup\{a_j^{(1)}(\pi'),a_j^{(2)}(\pi')\}_{j=1}^{2(d-\nu)}.$$

Consider $B_j\colon\mathbb{R}^2\to\mathbb{R}^2$ defined by $(B_jx)_i := \langle x, a_j^{(i)}\rangle \quad (i=1,2)$ which is injective. Let $K>0$ be such that $\|B_jx\|_\infty \leq K\|x\|_\infty\ \forall\ x,j$.

By Hölder's inequality,

$$\begin{aligned}
&\sum_{(k,\ell)\in D(\pi)\cap\mathbb{N}_n^2}\prod_{j=1}^d u(\langle a_j^{(1)},(k,\ell)\rangle)u(\langle a_j^{(2)},(k,\ell)\rangle)\\
&\le \prod_{j=1}^d\Big(\sum_{(k,\ell)\in D(\pi)\cap\mathbb{N}_n^2} u(\langle a_j^{(1)},(k,\ell)\rangle)^d u(\langle a_j^{(2)},(k,\ell)\rangle)^d\Big)^{\frac{1}{d}}\\
&= \prod_{j=1}^d\Big(\sum_{(k,\ell)\in B_j(D(\pi)\cap\mathbb{N}_n^2)} u(k)^d u(\ell)^d\Big)^{\frac{1}{d}}\\
&\le \sum_{(k,\ell)\in\mathbb{N}_{Kn}^2} u(k)^d u(\ell)^d\\
&= a_d(Kn)^2\\
&\ll a_d(n)^2. \quad ☑
\end{aligned}$$

Proof of theorem 2.2 For any $0\le\nu\le d$, fix $A_0,\dots,A_d,B_0,\dots,B_d\in\mathcal{A}$ so that $A_j=B_j\ \forall\ 0\le j\le d,\ j\ne\nu$, then

$$\begin{aligned}
\sum_{k=0}^n &|m(\bigcap_{j=1}^d T^{-jk}A_j)-m(\bigcap_{j=1}^d T^{-jk}B_j)|\\
&= \sum_{k=0}^n m(\bigcap_{j=1,\ j\ne\nu}^d T^{-jk}A_j\cap T^{-\nu k}(A_\nu\Delta B_\nu))\\
&\le \int_{A_\nu\Delta B_\nu}\psi_n^{(\nu)}dm\\
&\le \sqrt{m(A_\nu\Delta B_\nu)}\sqrt{\int_\Omega(\psi_n^{(\nu)})^2dm}\\
&\le M\sqrt{m(A_\nu\Delta B_\nu)}a_d(n). \quad ☑
\end{aligned}$$

§3 Multiple rational weak mixing.

Let $d\in\mathbb{N}$. We'll call the `CEMPT` $(X,\mathcal{B},m,T)$ *d-rationally weakly mixing along* $\mathfrak{K}\subset\mathbb{N}$ if it is d-rationally ergodic and

• $\exists\, u_d(n) > 0$ so that the normalizing constants are given by $a_d(n) := \sum_{k=0}^{n-1} u_d(n) \uparrow\ \infty$ where

$$\frac{1}{a_d(n)} \sum_{k=0}^{n-1} \left| m\Big(\bigcap_{j=0}^{d} T^{-(jk+r_j)} B_j\Big) - \prod_{j=0}^{d} m(B_j) u_d(k) \right| \xrightarrow[n\to\infty,\ n\in\mathfrak{K}]{} 0$$

Rational weak mixing (i.e. 1-rational weak mixing along $\mathbb{N}$) was introduced in [3].

Recall that the `CEMPT` $(X, \mathcal{B}, m, T)$ is *pointwise dual ergodic* if there are constants $a_n(T) > 0$ so that

$$\text{(DK)} \qquad \frac{1}{a_n(T)} \sum_{k=1}^{n} \widehat{T}^k f \xrightarrow[n\to\infty]{} \int_X f dm \ \text{ a.e. } \ \forall\ f \in L^1(m).$$

Here, $\widehat{T} : L^1(m) \to L^1(m)$ is the *transfer operator* defined by $\widehat{T} f := \frac{d\nu_f \circ T^{-1}}{dm}$ where $\nu_f(A) := \int_X f dm$. It satisfies

$$\int_X \widehat{T} f \cdot g dm = \int_X f \cdot T g dm \ \forall\ f \in L^1(m),\ g \in L^\infty(m).$$

First return time and induced transformation.

Suppose $(X, \mathcal{B}, m, T)$ is a `CEMPT` and let $\Omega \in \mathcal{B}$, $m(\Omega) > 0$, then m–a.e. point of Ω returns to Ω under iterations of T. The *return time* function to Ω, defined for $x \in \Omega$ by $\varphi_\Omega(x) := \min\{n \geq 1 : T^n x \in \Omega\}$ is finite m–a.e. on Ω.

The *induced transformation* on Ω is defined by $T_\Omega x = T^{\varphi_\Omega(x)} x$.

In case $m(\Omega) < \infty$, then $(\Omega, \mathcal{B} \cap \Omega, T_\Omega, m_\Omega)$ is an `EPPT` (ergodic, probability preserving transformation).

Proposition 3.1

Let $(X, \mathcal{B}, m, T)$ be an exact, pointwise dual ergodic `CEMPT` *and suppose that $\exists\ \Omega \in \mathcal{F}_+$ & $\alpha \subset \mathcal{B} \cap \Omega$ a one-sided T_Ω-generator for $\mathcal{B} \cap \Omega$ so that*

(i) the first return time $\varphi_\Omega : \Omega \to \mathbb{N}$ is α-measurable &

(ii) there exist $d \in \mathbb{N}$, $0 < \gamma < \frac{1}{d}$ and $b : \mathbb{R}_+ \to \mathbb{R}_+$, a $\frac{1}{\gamma}$-regularly varying function, so that

$$\texttt{(LLT)} \quad b(n) \widehat{T}_\Omega^n (1_{A \cap [\varphi_n = k_n]}) \xrightarrow[n\to\infty,\ \frac{k_n}{b(n)} \to x]{L^\infty(m_\Omega)} m_\Omega(A) f_\gamma(x)$$

$$\forall \text{ cylinders } A \ \&\ x \in \mathbb{R}_+,$$

where f_γ is the probability density function of the normalized, positive γ-stable random variable, then

$$(\text{✐})\quad \frac{1}{a_d(n)}\sum_{k=1}^{n}|\widehat{T}^{k+r_1}(1_{A_1}\widehat{T}^{k+r_2}(1_{A_2}\dots\widehat{T}^{k+r_d}(1_{A_d})\dots)-\prod_{i=1}^{d}m(A_i)u_k^d| \xrightarrow[n\to\infty]{} 0 \quad a.e.\ \ \forall\ A_1,\dots,A_d\in\mathcal{B}.$$

and, in particular, $(X,\mathcal{B},m,T)$ is d-rationally weakly mixing.

Remark. Suitable:

- `AFN` maps, and towers over `AFU maps` as in [6],
- towers over `Gibbs-Markov maps` as in [5]

satisfy the assumptions of proposition 3.1.

Proof

Let $a(n) := b^{-1}(n),\ u_n := \frac{\gamma a(n)}{n}$, then

$$a_n(T)\sim a(n)\sim\sum_{k=1}^{n}u_k.$$

We'll use pointwise dual ergodicity to establish rational weak mixing. To this end, we show first (as in [11]) that (`LLT`) implies that :

$$(\text{⚽})\quad \varliminf_{n\to\infty}\frac{1}{u_n}\widehat{T}^n 1_A\ \geq m(A)\ \text{ a.e. for } A\subset\Omega \text{ a union of cylinders.}$$

Proof of (⚽)

As in [11] (see also [3]):

$$\begin{aligned}\widehat{T}^n 1_A &= \sum_{k=1}^{n}\widehat{T}_\Omega^k 1_{A\cap[\varphi_k=n]}\\ &\geq \sum_{1\leq k\leq n,\ x_{k,n}\in[c,d]}\widehat{T}_\Omega^k 1_{A\cap[\varphi_k=x_{k,n}b(k)]}\quad\text{where } x_{k,n}:=\tfrac{n}{b(k)}\\ &\overset{(\text{LLT})}{\sim}\sum_{1\leq k\leq n,\ x_{k,n}\in[c,d]}\tfrac{f(x_{k,n})}{b(k)}m(A).\end{aligned}$$

By the γ-regular variation of a,

$$\frac{1}{b(k)}\sim\ \frac{\gamma a(n)}{n}\cdot\frac{x_{k,n}-x_{k+1,n}}{x_{k,n}^{\gamma}}.$$

Therefore

$$\begin{aligned}\frac{1}{u_n}\sum_{1\le k\le n,\ x_{k,n}\in[c,d]}\frac{f(x_{k,n})}{b(k)} &\approx \sum_{1\le k\le n,\ x_{k,n}\in(c,d)}\frac{(x_{k,n}-x_{k+1,n})}{x_{k,n}^{\gamma}}f(x_{k,n})\\ &\xrightarrow[n\to\infty]{} \int_{[c,d]}\frac{f(x)dx}{x^{\gamma}}\\ &= \mathbb{E}(1_{[c,d]}(Z_\gamma)Z_\gamma^{-\gamma})\\ &\xrightarrow[c\to 0+,\ d\to\infty]{} 1.\ \text{☑⚽}\end{aligned}$$

Proof of (✎)

Let $A\subset\Omega$ be a finite unions of cylinders. By (DK), (⚽) and proposition 3.3 in [3],

$$\frac{1}{a(n)}\sum_{k=1}^{n}|\widehat{T}^{n+r}1_A - m(A)u_n| \xrightarrow[n\to\infty]{} 0 \ \text{ a.e.}$$

whence by proposition 3.1 in [3], for a.e. $x\in\Omega$, there is a subset $K=K_x\subset\mathbb{N}$ of full density so that

$$\frac{1}{u_n}\widehat{T}^{n+r}1_A(x) \xrightarrow[n\to\infty,\ n\in K_x]{} m(A).$$

Let $C\subset\Omega$ be compact. As in [3], $\exists$ finite unions of cylinders $A_n\supset A_{n+1}\supset C$ so that $m(A_n)\downarrow m(C)$ as $n\to\infty$. Thus, for a.e. $x\in\Omega$, there is a subset $K=K_x\subset\mathbb{N}$ of full density so that

$$\overline{\lim_{n\to\infty,\ n\in K_x}}\frac{1}{u_n}\widehat{T}^{n+r}1_C(x)\le m(C),$$

whence, again by (DK) and propositions 3.1 & 3.3 in [3],

$$\frac{1}{u_n}\widehat{T}^{n+r}1_C(x) \xrightarrow[n\to\infty,\ n\in K_x]{} m(C).$$

Let $B\in\mathcal{B}(\Omega)$. Again as in [3], there are compact sets $C_n\subset C_{n+1}\subset B$ so that $m(C_n)\uparrow m(B)$ as $n\to\infty$. Thus, for a.e. $x\in\Omega$, there is a subset $K=K_x\subset\mathbb{N}$ of full density so that

$$\underline{\lim_{n\to\infty,\ n\in K_x}}\frac{1}{u_n}\widehat{T}^{n+r}1_B(x)\ge m(B),$$

whence, (as above) for a possibly smaller $K=K_x\subset\mathbb{N}$ of full density,

$$\frac{1}{u_n}\widehat{T}^{n+r}1_B(x) \xrightarrow[n\to\infty,\ n\in K_x]{} m(B).$$

It follows that for $A_1,\dots,A_d\in\mathcal{B}(\Omega)$, for a.e. $x\in\Omega$, there is a subset $K=K_x\subset\mathbb{N}$ of full density so that

$$\frac{1}{u_n^d}\widehat{T}^{k+r_1}(1_{A_1}\widehat{T}^{k+r_2}(1_{A_2}\dots\widehat{T}^{k+r_d}(1_{A_d})\dots)\xrightarrow[n\to\infty,\ n\in K_x]{}\prod_{i=1}^d m(A_i).$$

The index of regular variation of u_n^{-d} is $d\gamma\in(0,1)$ so, again by propositions 3.1 and 3.3 in [3],

$$\frac{1}{a_d(n)}\sum_{k=1}^n|\widehat{T}^{k+r_1}(1_{A_1}\widehat{T}^{k+r_2}(1_{A_2}\dots\widehat{T}^{k+r_d}(1_{A_d})\dots)-\prod_{i=1}^d m(A_i)u_k^d|$$

$$\xrightarrow[n\to\infty]{}0\ \text{ a.e.}. \quad \boxtimes$$

§4 Special semiflows

Given a σ-finite measure space $(X,\mathcal{B},m)$ we denote

$$\mathtt{MPT}(X,\mathcal{B},m):=\{\text{measure preserving transformations of }(X,\mathcal{B},m)\}.$$

Note that these transformations are not necessarily invertible. In this section, we consider, for S a finite set and $\kappa\in\mathbb{N}$, measure preserving semiflows $\Psi:\mathbb{R}_+\to\mathtt{MPT}(X,\mathcal{B},m)$ where

$X=\{(x,n,t)\in\Omega\times\mathbb{Z}^\kappa\times\mathbb{R}_+:\ 0\le x<h(x)\}$ where

$\Omega\subset S^{\mathbb{N}}$ a transitive $\mathtt{SFT}$, $h:\Omega\to\mathbb{R}_+$ Hölder;

$\mathcal{B}=\mathcal{B}(X)$, $m(A\times B\times C)=\mu(A)\#(B)\mathtt{Leb}\,(C)$

where $\#$ is counting measure, $\mu\in\mathcal{P}(\Omega)$ is Gibbs as in [19], [10];

$\Psi_t(x,n,y)=(T^nx,n+\phi_n(x),y+t-h_n(x))$

where $\phi:\Omega\to\mathbb{Z}^\kappa$ is continuous, $T=\mathtt{Shift}$ & $n=n_t(x,y)$ is s.t.

$$h_n(x):=\sum_{k=0}^{n-1}h(T^kx)\le y+t<h_{n+1}(x).$$

Pointwise dual ergodicity. Suppose that, Ψ is ergodic, equivalently T_ϕ is ergodic, or ϕ is *non-arithmetic* in the sense that $\nexists$ solution to

$\phi=k+g-g\circ T$, $g:\Omega\to\mathbb{Z}^\kappa$ measurable,

$k:\Omega\to\mathbb{K}$ measurable, where $\mathbb{K}$ is a proper subgroup of $\mathbb{Z}^\kappa$.

By the central limit theorem in [12],

$$(\mathtt{CLT})\qquad \frac{\phi_n}{\sqrt{n}}\xrightarrow[n\to\infty]{\text{distribution}}X$$

where X is a globally supported, centered Gaussian random variable on $\mathbb{R}^\kappa$.

As in [4], T_ϕ is dissipative for $\kappa \geq 3$ and pointwise dual ergodic for $\kappa = 1, 2$ with

$$a_n(T_\phi) \sim \sum_{k=0}^{n} u_k(T_\phi) \ \sim \begin{cases} 2\sqrt{n} f_X(0) & \kappa = 1; \\ \log n f_X(0) & \kappa = 2. \end{cases}$$

Analogously to Proposition 2.2 in [4], $(X, \mathcal{B}, m, \Psi)$ is dissipative for $\kappa \geq 3$ and pointwise dual ergodic (as a flow) for $\kappa = 1, 2$ with

$$a_n(\Psi) \sim \varkappa^{\frac{\kappa}{2}-1} a_n(T_\phi)$$

where $\varkappa = \int_\Omega h dP \in \mathbb{R}_+$: namely

$$\frac{1}{a_n(\Psi)} \int_0^n \widehat{\Psi}_t(F) dt \xrightarrow[n\to\infty]{\text{a.e.}} \int_X F dm \ \forall \ F \in L^1(m).$$

Proposition 4.1 **(Exactness)** *Suppose that the function $(h, \phi) : \Omega \to \mathbb{G} := \mathbb{R} \times \mathbb{Z}^\kappa$ is* `non-arithmetic` *in the sense that $\nexists$ solution to*

$$(h, \phi) = k + g - g \circ T, \quad g : \Omega \to \mathbb{G} \quad \textit{measurable},$$
$$k : \Omega \to \mathbb{K} \quad \textit{measurable, where } \mathbb{K} \textit{ is a proper subgroup of } \mathbb{G},$$

then $(X, \mathcal{B}, m, \Psi_t)$ is an exact endomorphism $\forall \ t > 0$.

Proof The assumption is equivalent to the ergodicity of

$$(\Omega \times \mathbb{G}, \mu \times \texttt{Leb} \times \#, T_{(h,\phi)})$$

which entails (characterizes) the exactness of Ψ. ☑

In this case, for each $t > 0$, $(X, \mathcal{B}, m, \Psi_t)$ is pointwise dual ergodic (as a transformation) for $\kappa = 1, 2$ with

$$a_n(\Psi_t) \sim \varkappa^{\frac{\kappa}{2}-1} a_{tn}(T_\phi),$$

namely

$$\texttt{(PDE)} \qquad \frac{1}{a_n(\Psi_t)} \sum_{k=0}^{n} \widehat{\Psi}_{tk}(F) \xrightarrow[n\to\infty]{\text{a.e.}} \int_X F dm \ \forall \ F \in L^1(m).$$

Rational weak mixing.

If ϕ is *aperiodic* in the sense that $\nexists$ solution to

$$\phi = k + g - g \circ T, \quad g : \Omega \to \mathbb{Z}^\kappa \quad \text{measurable},$$
$$k : \Omega \to \mathbb{K} \ \ \text{measurable, where } \mathbb{K} \text{ is a proper coset of } \mathbb{Z}^\kappa,$$

then by theorem C* in [13] (which in this case follows from the local limit theorem in [12]), for $A \subset \Omega$ a cylinder set:

$$(\mathtt{LLT}) \quad \widehat{T}^n(1_{A\cap[\phi_n=\lfloor t_n\rfloor]}) \sim \frac{f_X(\frac{t_n}{\sqrt{n}})\mu(A)}{n^{\frac{\kappa}{2}}} \text{ as } n\to\infty \;\&\; t_n = O(\sqrt{n}).$$

In particular,

$$\widehat{T}_\phi(1_{A\times\{0\}}) = \widehat{T}^n(1_{A\cap[\phi_n=0]}) \sim \frac{\mu(A)f_X(0)}{n^{\frac{\kappa}{2}}} =: u_n(T_\phi)\mu(A) \text{ as } n\to\infty.$$

It follows from proposition 3.1 that $(\Omega\times\mathbb{Z}^\kappa, \mathcal{B}(\Omega\times\mathbb{Z}^\kappa), \mu\times\#, T_\phi)$ is rationally weakly mixing when $\kappa = 1, 2$.

Proposition 4.2 (`RWM` of special semiflows)

Suppose that the function $(h,\phi) : \Omega \to \mathbb{R}\times\mathbb{Z}^\kappa$ is `aperiodic` *in the sense that $\nexists$ solution to*

$$(h,\phi) = k + g - g\circ T, \quad g : \Omega\to\mathbb{R}\times\mathbb{Z}^\kappa \quad \textit{measurable},$$
$$k : \Omega \to \mathbb{K} \quad \textit{measurable, where } \mathbb{K} \textit{ is a proper coset of } \mathbb{R}\times\mathbb{Z}^\kappa,$$

then for each $t > 0$, $(X, \mathcal{B}, m, \Psi_t)$ is rationally weakly mixing.

`RWM` entails a kind of "density local limit theorem" and we begin the proof of proposition 4.2 with a one-sided version of (`LLT`) for the flow transformations.

Lemma 4.3: (Lower local limit)

Suppose that the function $(h,\phi) : \Omega \to \mathbb{R}\times\mathbb{Z}^\kappa$ is `aperiodic` *in the sense that $\nexists$ solution to*

$$(h,\phi) = k + g - g\circ T, \quad g : \Omega\to\mathbb{R}\times\mathbb{Z}^\kappa \quad \textit{measurable},$$
$$k : \Omega \to \mathbb{K} \quad \textit{measurable, where } \mathbb{K} \textit{ is a proper coset of } \mathbb{R}\times\mathbb{Z}^\kappa,$$

then for $A \subset \Omega$ a cylinder set and $I \subset [0, \min h]$ an interval,

$$(\mathtt{LLL}) \qquad \varliminf_{t\to\infty} t^{\frac{\kappa}{2}}\widehat{\Psi}_t(1_A\otimes 1_{\{0\}}\otimes 1_I)(\omega,0,y) \ge \varkappa^{\frac{\kappa}{2}-1} f_X(0)\mu(A)|I|.$$

Proof of (`LLL`)

We have that

$$\widehat{\Psi}_t(1_A\otimes 1_{\{0\}}\otimes 1_I)(\omega,y,0) = \sum_{n=0}^{\infty} \widehat{T}^n(f 1_{[\phi_n=0,\ h_n\in I+t-y]})(\omega)$$

and so it suffices to prove

$$(\boxtimes) \qquad \lim_{M\to\infty}\lim_{t\to\infty} t^{\frac{\kappa}{2}} \sum_{n=\frac{t}{\varkappa}\pm M\sqrt{t}} \widehat{T}^n(1_{A\cap[\phi_n=0,\ h_n\in I+t-y]})$$
$$= \varkappa^{\frac{\kappa}{2}-1} f_X(0)\mu(A)|I|.$$

Proof of (⊠)

By the central limit theorem for (h, ϕ),

$$\frac{1}{\sqrt{n}}(\phi_n, h_n - \varkappa n) \xrightarrow[n\to\infty]{\text{distribution}} Z$$

where $Z = (X, Y)$ is non-singular normal with

- X centered, non-singular normal on $\mathbb{R}^\kappa$,
- Y centered normal on $\mathbb{R}$.

By the LLT for (h, ϕ),

$$n^{\frac{\kappa+1}{2}}\widehat{T}^n(1_A 1_{[\phi_n=0,\ h_n\in I+n\varkappa+x_n\sqrt{n}]}) \underset{n\to\infty,\ |x_n|\le M}{\approx} f_Z(0, x_n)\mu(A)|I|.$$

Fix $t,\ M > 0$, then

$$t = \varkappa n + x\sqrt{n} \ \text{ with } \ x = x_{n,t} \in [-M, M] \quad \iff$$

$$n = \frac{t}{\varkappa} - \frac{x}{\varkappa^{\frac{3}{2}}}\sqrt{t} \ \text{ \& in this case}$$

$$\widehat{T}^n(f 1_{[\phi_n=0,\ h_n\in I+t-y]}) \sim \frac{|I|}{n^{\frac{\kappa+1}{2}}} f_Z(0, x_{n,t})$$

as $t,\ n\to\infty,\ |x_{n,t}| \le M$.

It follows that for fixed $M > 0$ with $M' := \frac{M}{\varkappa^{\frac{3}{2}}}$,

$$\begin{aligned}
t^{\frac{\kappa}{2}} \sum_{n=\frac{t}{\varkappa}\pm M'\sqrt{t}} &\widehat{T}^n(f 1_{[\phi_n=0,\ h_n\in I+t-y]}) \\
&\underset{t\to\infty}{\sim} \varkappa^{\frac{\kappa}{2}} n^{\frac{\kappa}{2}} \sum_{n=\frac{t}{\varkappa}\pm M'\sqrt{t}} \widehat{T}^n(f 1_{[\phi_n=0,\ h_n\in I+t-y]}) \\
&\underset{t\to\infty}{\sim} \varkappa^{\frac{\kappa}{2}} \sum_{n=\frac{t}{\varkappa}\pm M'\sqrt{t}} \frac{|I|}{\sqrt{n}} f_Z(0, x_{n,t})
\end{aligned}$$

Now,

$$\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \sim \frac{1}{2n\sqrt{n}}$$

so

$$\begin{aligned}
x_{n,t} - x_{n+1,t} &= \frac{t - \varkappa n}{\sqrt{n}} - \frac{t - \varkappa(n+1)}{\sqrt{n+1)}} \\
&= \frac{\varkappa}{\sqrt{n+1}} + (t - \varkappa n)(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}) \\
&= \frac{\varkappa}{\sqrt{n}}(1 + O(\frac{1}{\sqrt{n}})).
\end{aligned}$$

Thus

$$\begin{aligned}
\sqrt{t}\sum_{n=\frac{t}{\varkappa}\pm M'\sqrt{t}} & \widehat{T}^n(f1_{[\phi_n=0,\ h_n\in I+t-y]}) \\
& \underset{t\to\infty}{\approx}\ \varkappa^{\frac{\kappa}{2}}\sum_{n=\frac{t}{\varkappa}\pm M'\sqrt{t}}\frac{|I|}{\sqrt{n}}f_Z(0,x_{n,t}) \\
& \underset{t\to\infty}{\approx}\ \varkappa^{\frac{\kappa}{2}-1}|I|\sum_{n=\frac{t}{\varkappa}\pm M'\sqrt{t}}(x_{n+1,t}-x_{n,t})f_Z(0,x_{n,t}) \\
& \xrightarrow[t\to\infty]{}\ \varkappa^{\frac{\kappa}{2}-1}|I|\int_{[-M',M']}f_Z(0,x)dx \\
& \xrightarrow[M'\to\infty]{}\varkappa^{\frac{\kappa}{2}-1}|I|f_X(0).\quad ☑
\end{aligned}$$

Proof of proposition 4.2 This follows from pointwise dual ergodicity and (LLL) via proposition 3.3 of [3]. ☑

Remark. A stronger version of lemma 4.3 would be the `local limit theorem`:

For $A\subset\Omega$ a cylinder set and $I\subset[0,\min h]$ an interval,

$$(\mathtt{LLT})\qquad \lim_{t\to\infty}t^{\frac{\kappa}{2}}\widehat{\Psi}_t(1_A\otimes 1_{\{0\}}\otimes 1_I)(\omega,0,y)=\varkappa^{\frac{\kappa}{2}-1}f_X(0)\mu(A)|I|.$$

It is not hard to show that special semiflows satisfying (`LLT`) enjoy the stronger property of `Krickeberg mixing` (as in [17]). Note that here, in the notation of lemma 4.3, (`LLT`) is equivalent to

$$(🔔)\qquad \overline{\lim_{M\to\infty}}\ \overline{\lim_{t\to\infty}}\ t^{\frac{\kappa}{2}}\sum_{n\ge 1,\ |n-\frac{t}{\varkappa}|\ge M\sqrt{t}}\widehat{T}^n(1_{[\phi_n=0]})=0.$$

We do not know whether this necessarily holds under the assumptions of lemma 4.3, but we'll see in the next section that the geodesic flows of Abelian covers of compact hyperbolic surfaces have this $\overline{\lim}$ finite when we show (🚂) (on page 19).

For work on (`LLT`) for flows, see [16] & [24].

§5 HYPERBOLIC GEODESIC FLOWS

Definitions. The *hyperbolic plane* is $\mathbb{H}:=\{z=u+iv\in\mathbb{C}:|z|<1\}$ equipped with the arclength element $ds(u,v):=\frac{2\sqrt{du^2+dv^2}}{1-u^2-v^2}$ and the area element $dA(u,v):=\frac{4dudv}{(1-u^2-v^2)^2}$.

The *hyperbolic distance* between $x,y\in\mathbb{H}$ is

$$\rho(x,y):=\ \inf\{\int_\gamma ds:\ \gamma\text{ is an arc joining }x\text{ and }y\}=2\tanh^{-1}\frac{|x-y|}{|1-\overline{x}y|}.$$

This inf is achieved by an arc of a *geodesic* in $(\mathbb{H}, \rho)$. Note that $\mathbb{H}$ is also known as the `disk model` for the hyperbolic plane. These geodesics are diameters of $\mathbb{H}$ and circles orthogonal to $\partial\mathbb{H}$.[1]

The isometries $\mathtt{Isom}\,(\mathbb{H}, \rho)$ of $(\mathbb{H}, \rho)$ are the Möbius transformations and their complex conjugates.

If g is an isometry of $\mathbb{H}$, then $A \circ g \equiv A$.

The space of *line elements* of $\mathbb{H}$ is $UT(\mathbb{H}) = \mathbb{H} \times \mathbb{T}$.

The *geodesic flow transformations* φ^t are defined on $\mathbb{H} \times \mathbb{T}$ as follows. To each line element ω there corresponds a unique directed geodesic passing through $x(\omega)$ whose directed tangent at $x(\omega)$ makes an angle $\theta(\omega)$ (with the radius $(0, 1)$).

If $t > 0$, the point $x(\varphi^t\omega)$ is the unique point on the geodesic at distance t from $x(\omega)$ in the direction of the geodesic, and if $t < 0$, the point $x(\varphi^t\omega)$ is the unique point on the geodesic at distance $-t$ against the direction of the geodesic.

The angle $\theta(\varphi^t\omega)$ is the angle made by the directed tangent to the geodesic at the point $x(\varphi^t\omega)$.

There is an important involution $\chi : \mathbb{H} \times \mathbb{T} \to \mathbb{H} \times \mathbb{T}$, of direction reversal: $x(\chi\omega) = x(\omega)$ and $\theta(\chi\omega) = \theta(\omega) + \pi$. Here $\omega = (x, \theta) = (x(\omega), \theta(\omega))$.

The isometries act on $\mathbb{H} \times \mathbb{T}$ (as differentiable maps) by

$$g(\omega) = (g(x(\omega)), \theta(\omega) + \arg g'(x(\omega))$$

and it is not hard to see that $\chi g = g\chi$ and $\varphi^t g = g\varphi^t$.

Both the geodesic flow, the involution and the isometries preserve the measure

$dm(x, \theta) = dA(x)d\theta$ on $\mathbb{H} \times \mathbb{T}$.

Let Γ be a discrete subgroup of $\mathtt{Isom}\,(\mathbb{H})$ (aka *Fuchsian group*), then $M = \mathbb{H}/\Gamma$ is an hyperbolic surface and any hyperbolic surface is isometric to one of this form.

The space of line elements of $M = \mathbb{H}/\Gamma$ is $UT(M) := M \times \mathbb{T} = (\mathbb{H} \times \mathbb{T})/\Gamma$ and the geodesic flow transformations on $UT(M)$ are defined by

$$\varphi_t^M \Gamma(\omega) := \Gamma\varphi^t(\omega).$$

Let $\pi_\Gamma : \mathbb{H} \to M$, $\overline{\pi}_\Gamma : \mathbb{H} \times \mathbb{T} \to UT(M)$ be the projections $\pi_\Gamma(z) = \Gamma z$, $\overline{\pi}_\Gamma(\omega) = \Gamma\omega$, and let F be a *fundamental domain* for Γ in $\mathbb{H}$, e.g.

$$F^o := \{x \in \mathbb{H} : \rho(y, x) < \rho(\gamma(y), x)\ \forall\ \gamma \in \Gamma \setminus \{e\}\}, \quad y \in \mathbb{H},$$

[1] It is sometimes convenient to consider the conformally equivalent `upper half plane model` $\widehat{\mathbb{H}} := \{x + iy :\ x, y \in \mathbb{R},\ y > 0\}$ where the geodesics are vertical lines and circles orthogonal to $\mathbb{R}$.

then π_Γ and $\overline{\pi}_\Gamma$ are 1-1 on F and $F\times\mathbb{T}$, and so the measures $A_{|F}$ and $m_{|F}$ induce measures A_Γ and m_Γ on $M=\mathbb{H}/\Gamma$ and $UT(M)=\mathbb{H}/\Gamma\times\mathbb{T}$ respectively.

Basics. It is known that for $M=\mathbb{H}/\Gamma$,

- φ^M is either totally dissipative, or conservative and ergodic (E. Hopf [14]),

- φ^M is conservative iff $a_\Gamma(t) := \displaystyle\sum_{\gamma\in\Gamma,\ \rho(x,\gamma(x))\le t} e^{-\rho(x,\gamma(x))} \xrightarrow[t\to\infty]{} \infty$

(E. Hopf [15] & M. Tsuji [23]).

Moreover, any conservative φ^M is:

- rationally ergodic with return sequence $\propto\ a_\Gamma(t)$ ([8], see also [2] chapter 7);
- weakly mixing ([20]). Note that a flow is weakly mixing iff all its transformations are ergodic.

All transformations of a rationally ergodic, weakly mixing flow are necessarily rationally ergodic.

Abelian covers of compact surfaces.

Let $M=\mathbb{H}/\Gamma$ be a compact, hyperbolic surface, let $\varphi^M: UT(M)\to UT(M)$ denote the geodesic flow and let $\chi: UT(M)\to UT(M)$ be the involution of direction reversal.

Now let $\kappa\ge 1$ & let $V=V^{(\kappa)}$ be a $\mathbb{Z}^\kappa$-cover of M that is V is a complete hyperbolic surface equipped with a covering map $p:V\to M$ so that $\exists$ a monomorphism $\gamma:\mathbb{Z}^\kappa\to \mathtt{Isom}\,(V^{(\kappa)})$, such that for $y\in V,\ p^{-1}\{p(y)\}=\{\gamma(n)y:\ n\in\mathbb{Z}^\kappa\}$.

Rees showed in [19] that $\varphi^{V^{(\kappa)}}$ is conservative when $\kappa=1,2$ and dissipative when $\kappa\ge 3$.

In this section we prove

Theorem 5.1 *The geodesic flow transformations* $\varphi_t^{V^{(\kappa)}}$ $(t>0)$ *are*

- *rationally weakly mixing when* $\kappa=1,2$*;*
- 2*-recurrent when* $\kappa=1$ *and* 2*-dissipative when* $\kappa=2$*.*

Proof of rational weak mixing

Let $M=\mathbb{H}/\Gamma$ be a compact, hyperbolic surface (with Γ is the corresponding, cocompact, Fuchsian group) and let $\varphi_M: UT(M)\to UT(M)$ denote the geodesic flow on $UT(M)$ (the unit tangent bundle) and let $\chi: UT(M)\to UT(M)$ be the involution of direction reversal.

As φ^M is an Anosov flow, by Bowen's theorem ([9]), there is a special flow $\Phi : \mathbb{R} \to \texttt{PPT}(Y, \mathcal{C}, \nu)$ and $\pi : Y \to UT(M)$ a continuous, measure theoretic isomorphism satisfying $\varphi^M \circ \pi = \pi \circ \Phi$. Here $\texttt{PPT}(Y, \mathcal{C}, \nu)$ denotes the collection of invertible probability preserving transformations of the probability space $(Y, \mathcal{C}, \nu)$ equipped with the weak operator topology.

Here:

$Y = \{(x, t) \in \Omega \times \mathbb{R}_+ :\ 0 \le x < h(x)\}$ where

$\Omega \subset S^{\mathbb{Z}}$ a transitive $\texttt{SFT}$, $h : \Omega \to \mathbb{R}_+$ Hölder;

$\mathcal{C} = \mathcal{B}(Y), \quad \nu(A \times B) = c^{-1}\mu(A)\texttt{Leb}\,(B)$

$\mu \in \mathcal{P}(\Omega)$ Gibbs as in [10]; $c := \int_\Omega h d\mu$ & $\Phi_t(x, y) = (T^n x, y + t - h_n(x))$

where T is the shift and $n = n_t(x, y)$ is so that

$$h_n(x) := \sum_{k=0}^{n-1} h(T^k x) \le y + t < h_{n+1}(x).$$

By Rees' refinement, (Ω, T, μ), h & π can be chosen so that

- S is a finite, symmetric generator set of Γ and the elements of Ω code the geodesics in $M = \mathbb{H}/\Gamma$,
- (Ω, T, μ) is topologically mixing,
- $h(\dots, x_{-1}, x_0, x_1, \dots) = h(x_1, x_2, \dots)$ and
- $\chi(\pi\Sigma) = \pi\Sigma$.

Now let $\kappa \ge 1$ & let V be a $\mathbb{Z}^\kappa$-cover of M that is V is a complete hyperbolic surface equipped with a covering map $p : V \to M$ so that there exists a monomorphism $\gamma : \mathbb{Z}^\kappa \to \texttt{Isom}\,(V)$, such that for $y \in V$, $p^{-1}\{p(y)\} = \{\gamma(n)y :\ n \in \mathbb{Z}^\kappa\}$.

The corresponding tangent map, also denoted $p : UT(V) \to UT(M)$ is equivariant with the geodesic flows and their direction reversal involutions.

We have that $V^{(\kappa)} \cong \mathbb{H}/\Gamma_0$ where the corresponding Fuchsian group $\Gamma_0 = \texttt{Ker}\,\Theta$ for $\Theta : \Gamma \to \mathbb{Z}^\kappa$ a surjective homomorphism.

The corresponding $\mathbb{Z}^\kappa$-extension of $\Phi : \mathbb{R} \to \texttt{PPT}(Y, \mathcal{C}, \nu)$ is the special flow $\Psi : \mathbb{R} \to \texttt{MPT}(X, \mathcal{B}, m)$ where

$X = \{(x, n, t) \in \Omega \times \mathbb{Z} \times \mathbb{R}_+ :\ 0 \le x < h(x)\}$,

$m(A \times B \times C) = \mu(A)\texttt{Leb}\,(B)\#(C)$ where $\#$ is counting measure,

$\Psi_t(x, y, z) = (\Phi_t(x, y), z + \phi_n(x))$ where

$\Phi_t(x, y) = (T^n x, y + t - h_n(x))$ & $\phi(\omega_1, \omega_2, \dots) = \Theta(\omega_1)$.

There is a continuous, measure theoretic isomorphism $\Pi : X \to TV$ satisfying

$$p \circ \Pi \equiv \pi, \quad \varphi_V \circ \pi = \pi \circ \Psi \;\;\&\;\; \Pi(x,t,n) := \gamma(n)\Pi(x,t,0).$$

As in [21], V is *homologically full* in the sense that as $t \to \infty$, $\exists$ exponentially many closed geodesics of length $\le t$ in each homology class.

Therefore, by the lemma in [22] the function $(h,\phi) : \Omega \to \mathbb{R} \times \mathbb{Z}^\kappa$ is `aperiodic` in the sense that $\nexists$ solution to

$$\begin{aligned}&(h,\phi) = k + g - g \circ T, \quad g : \Omega \to \mathbb{R} \times \mathbb{Z}^\kappa \;\; \text{measurable},\\ &k : \Omega \to \mathbb{K} \;\; \text{measurable, where } \mathbb{K} \text{ is a proper coset of } \mathbb{R} \times \mathbb{Z}^\kappa.\end{aligned}$$

Thus $\Psi : \mathbb{R} \to \mathtt{MPT}(X, \mathcal{B}, m)$ is a two-sided version of a special semiflow satisfying the assumptions of proposition 4.2 and rational weak mixing follows. ☑

Proof of 2-recurrence & 2-dissipation

For $x \in \mathbb{H}$, and $\epsilon > 0$, set

$$N_\rho(x,\epsilon) = \{y \in \mathbb{H} : \rho(x,y) < \epsilon\}, \quad \Delta(x,\epsilon) := N_\rho(x,\epsilon) \times \mathbb{T}.$$

To prove the theorem, we show first that
sets of form $\Delta = \Delta(x,\epsilon)$ are admissible and satisfy

(🚂) $$m(\Delta \cap \varphi_{V^{(d)}}^{-t}\Delta) \;\asymp\; \frac{1}{t^{\frac{d}{2}}}.$$

The lower bound in (🚂) follows from lemma 4.3 applied to the flow $(X, \mathcal{B}, m, \Psi)$ as above.

We now proceed to establish the upper bound. The following analytic geometry lemmas are valid for any Fuchsian group Γ with $X_\Gamma = UT(\mathbb{H}/\Gamma) = \mathbb{H}/\Gamma \times \mathbb{T}$.

Analytic geometry lemma I*:*

For $x \in X_\Gamma$ *&* $\epsilon > 0$ *small enough:*

(i) $$m(\Delta(x,\epsilon) \cap \phi_\Gamma^{-t}\Delta(x,\epsilon)) \;\ll\; \sum_{\gamma \in \Gamma,\; \rho(x,\gamma(x) = t \pm 2\epsilon} e^{-\rho(x,\gamma(x))}$$

(ii) $$m(\Delta(x,\epsilon) \cap \phi_\Gamma^{-t}\Delta(x,\epsilon)) \gg \sum_{\gamma \in \Gamma,\; \rho(x,\gamma(x) = t \pm \frac{\epsilon}{2}} e^{-\rho(x,\gamma(x))}.$$

Proof We have that

$$m(\Delta(x,\epsilon)\cap\varphi_\Gamma^{-s}\Delta(x,\epsilon)) = \int_{\Delta(x,\epsilon)} 1_{\Delta(x,\epsilon)}\circ\varphi_\Gamma^s dm_\Gamma$$
$$= \int_{N_\rho(x,\epsilon)} \Phi(s;z)dA(z)$$

where

$$\Phi(s;z) := \sum_{\gamma\in\Gamma}\int_{\mathbb{T}} 1_{\gamma N_\rho(x,\epsilon)\times\mathbb{T}}\circ\varphi_\Gamma^s(z,\theta)d\theta.$$

Set $\varphi_z(\omega) = \frac{z+\omega}{1+\overline{z}\omega}$. Using $\varphi_z\varphi^t = \varphi^t\varphi_z$, and $\varphi^t(0,\theta) = (\tanh\frac{t}{2}e^{2\pi i\theta},\theta)$, we have

$$\begin{aligned}\Phi(s;z) &= \sum_{\gamma\in\Gamma}\int_{\mathbb{T}} 1_{\gamma N_\rho(x,\epsilon)\times\mathbb{T}}\circ\varphi_\Gamma^s(z,\theta)d\theta\\ &= \sum_{\gamma\in\Gamma}\int_{\mathbb{T}} 1_{\varphi_z^{-1}\gamma N_\rho(x,\epsilon)\times\mathbb{T}}\circ\varphi_\Gamma^s(0,\theta)d\theta\\ &= \sum_{\gamma\in\Gamma}\int_{\mathbb{T}} 1_{\varphi_z^{-1}\gamma N_\rho(x,\epsilon)}(\tanh(\tfrac{s}{2})e^{2\pi i\theta})d\theta\\ &= \sum_{\gamma\in\Gamma}\int_{\mathbb{T}} 1_{N_\rho(\varphi_z^{-1}\gamma(x),\epsilon)}(\tanh(\tfrac{s}{2})e^{2\pi i\theta})d\theta\\ &= \sum_{\gamma\in\Gamma}|J(\varphi_z^{-1}\gamma(x),\epsilon)|\end{aligned}$$

where $J(w,\eta)\subset\mathbb{T}$ is the interval

$$J(w,\eta) := \{\theta\in\mathbb{T}:\ \tanh(\tfrac{s}{2})e^{2\pi i\theta}\in N_\rho(w,\eta)\}$$

and $|J(w,\eta)|$ is its length.

We have that $|J(w,\eta)|>0$ iff $\rho(0,w)=s\pm\eta$.

Thus $|J(\varphi_z^{-1}\gamma(x),\epsilon)|>0$ iff

$$\rho(z,\gamma(x)) = \rho(0,\varphi_z^{-1}\gamma(x)) = s\pm\epsilon$$

and

$$\Phi(s;z) = \sum_{\gamma\in\Gamma,\ \rho(z,\gamma(x))=s\pm\epsilon}|J(\varphi_z^{-1}\gamma(x),\epsilon)|.$$

Next, for $w\in\mathbb{H}$ & $\eta>0$, we consider the angle interval subtended by $N_\rho(w,\eta)$ at $0\notin N_\rho(w,\eta)$,

$$\Lambda(w,\eta) := \{\theta\in[0,2\pi] : \exists\ r\in(0,1)\ \ni\ \rho(w,re^{i\theta})<\eta\}.$$

We note that

$$\Lambda(w,\eta)=\{\theta\in[0,2\pi]:\|\theta-\arg w\|<\sin^{-1}\Big(\frac{(1-|w|^2)\tanh\frac{\eta}{2}}{|w|(1-\tanh^2\frac{\eta}{2})}\Big)\},$$

where $\|\theta\|:=\theta\wedge(2\pi-\theta),\quad \theta\in[0,2\pi)$. This is because

$$N_\rho(w,\eta)=B\Big(\frac{(1-\delta^2)w}{1-\delta^2|w|^2},\frac{\delta(1-|w|^2)}{1-\delta^2|w|^2}\Big)$$

where $B(x,r)$ is the Euclidean ball of radius r and $\delta=\tanh\frac{\eta}{2}$. Thus as $|w|\to 1$ & $\eta\to 0$,

$$\begin{aligned}|\Lambda(w,\eta)|&=2\sin^{-1}\Big(\frac{(1-|w|^2)\tanh\frac{\eta}{2}}{|w|(1-\tanh^2\frac{\eta}{2})}\Big)\\&\sim\eta(1-|w|^2)\\&\sim\ \eta e^{-\rho(0,w)}\end{aligned}$$

Thus,

$$\begin{aligned}\Phi(s;z)&=\sum_{\gamma\in\Gamma,\ \rho(z,\gamma(x))=s\pm\epsilon}|J(\varphi_z^{-1}\gamma(x),\epsilon)|\\&\le\sum_{\gamma\in\Gamma,\ \rho(z,\gamma(x))=s\pm\epsilon}|\Lambda(\varphi_z^{-1}\gamma(x),\epsilon)|\\&\ll\sum_{\gamma\in\Gamma,\ \rho(z,\gamma(x))=s\pm\epsilon}e^{-\rho(z,\gamma(x))}\\&\le\sum_{\gamma\in\Gamma,\ \rho(x,\gamma(x))=s\pm 3\epsilon}e^{-\rho(x,\gamma(x))}.\end{aligned}$$

whence

$$\begin{aligned}m(\Delta(x,\epsilon)\cap\varphi_\Gamma^{-s}\Delta(x,\epsilon))&=\int_{N_\rho(x,\epsilon)}\Phi(s;z)dA(z)\\&\ll\sum_{\gamma\in\Gamma,\ \rho(x,\gamma(x))=s\pm 2\epsilon}e^{-\rho(x,\gamma(x))}.\quad ☑\text{(i)}\end{aligned}$$

Next, to establish (ii) note that $\exists\ \zeta>0$ so that

$$\rho(0,w)=s\pm\frac{3\eta}{4}\implies |J(w,\eta)|>\zeta|\Lambda(w,\eta)|.$$

Thus, for $z \in N_\rho(x, \frac{\epsilon}{4})$

$$\begin{aligned}
\Phi(s;z) &\geq \sum_{\gamma\in\Gamma,\ \rho(z,\gamma(x))=s\pm\frac{3\epsilon}{4}} |J(\varphi_z^{-1}\gamma(x),\epsilon)| \\
&\geq \zeta \sum_{\gamma\in\Gamma,\ \rho(z,\gamma(x))=s\pm\frac{3\epsilon}{4}} |\Lambda(\varphi_z^{-1}\gamma(x),\epsilon)| \\
&\gg \sum_{\gamma\in\Gamma,\ \rho(z,\gamma(x))=s\pm\frac{3\epsilon}{4}} e^{-\rho(z,\gamma(x))} \\
&\geq \sum_{\gamma\in\Gamma,\ \rho(x,\gamma(x))=s\pm\frac{\epsilon}{2}} e^{-\rho(x,\gamma(x))}
\end{aligned}$$

whence

$$\begin{aligned}
m(\Delta(x,\epsilon)\cap\varphi_\Gamma^{-s}\Delta(x,\epsilon)) &\geq \int_{N_\rho(x,\frac{\epsilon}{4})} \Phi(s;z)dA(z) \\
&\gg \sum_{\gamma\in\Gamma,\ \rho(x,\gamma(x))=s\pm\frac{\epsilon}{2}} e^{-\rho(x,\gamma(x))}. \quad \boxtimes\text{(ii)}
\end{aligned}$$

Analytic geometry lemma II*:*
For $x \in X_\Gamma$ & $\epsilon > 0$ *small enough:*

$$m(\bigcap_{j=0}^{p} \phi_\Gamma^{-\sum_{i=0}^{j} s_i}\Delta(x,\epsilon)) \leq M_p \prod_{k=1}^{p} m(\Delta(x,4\epsilon)\cap\phi_\Gamma^{-s_k}\Delta(x,4\epsilon))$$
$$\forall\ s_0 = 0,\ s_1,\ldots,s_p > 0.$$

Proof

For $\underline{\gamma} \in \Gamma^p$ (resp. $\underline{t} \in \mathbb{R}^p$) we denote its coordinates by γ_k (resp. t_k), $k = 1, ..., p$. Let

$$I_p = \{\underline{t} \in \mathbb{R}^p : 0 < t_1 < ... < t_p\}.$$

Let $\epsilon > 0$ be fixed and $\Delta = N \times \mathbb{T}$ as before, where $N = N_\rho(x,\epsilon)$. We assume ϵ to be sufficiently small.

First observe that

$$\begin{aligned}u(p,\underline{t}) &:= m(\bigcap_{j=0}^{p}\phi_\Gamma^{-t_j}\Delta)\\ &= \int_\Delta \prod_{j=1}^{p} 1_\Delta \circ \varphi_\Gamma^{t_j} dm_\Gamma\\ &= \sum_{\underline{\gamma}\in\Gamma^p}\int_\Delta \prod_{j=1}^{p} 1_{\gamma_j\Delta}\circ\varphi^{t_j}dm\\ &= \sum_{\underline{\gamma}\in\Gamma^p}\int_N\int_0^{2\pi}\prod_{j=1}^{p}1_{\gamma_jN\times\mathbb{T}}\circ\varphi^{t_j}(z,\theta)d\theta dA(z)\\ &= \int_N \psi_p(\underline{t},z)A(dz)\end{aligned}$$

where

$$\psi_p(\underline{t},z) := \sum_{\underline{\gamma}\in\Gamma^p}\int_0^{2\pi}\prod_{j=1}^{p}1_{\gamma_jN\times\mathbb{T}}\circ\varphi^{t_j}(z,\theta)d\theta.$$

Next,

$$\psi_p(t,z) = \sum_{\underline{\gamma}\in\Gamma^p}\int_0^{2\pi}\prod_{j=1}^{p}1_{\varphi_z^{-1}\gamma_jN}(\tanh t_je^{i\theta})d\theta$$

For $\underline{t}\in I_p$, , let $t_0=0$, let

$$s_{k+1} = t_{k+1}-t_k \quad (0\le k\le p-1)$$

and let

$$\Gamma_0(\underline{t}) := \{\underline{\gamma}\in\Gamma^p : \int_0^{2\pi}\prod_{j=1}^{p}1_{\varphi_z^{-1}\gamma_jN}(\tanh t_je^{i\theta})d\theta>0\}.$$

If $\underline{\gamma}=(\gamma_1,\dots,\gamma_p)\in\Gamma_0(\underline{t})$, then

$$\exists\ \theta\in[0,2\pi) \text{ with } \ \rho(\omega(\varphi_{t_k}(z,\theta)),\gamma_k(x))<\epsilon\ \forall\ 1\le k\le p$$

whence

$$\begin{aligned}\rho(\gamma_k(x),\gamma_{k+1}(x)) &= \rho(\omega(\varphi_{t_k}(z,\theta)),\omega(\varphi_{t_{k+1}}(z,\theta)))\pm 2\epsilon\\ &= s_k\pm 2\epsilon.\end{aligned}$$

Setting $w_0 = z = \gamma_0(x)$,

$$\begin{aligned}\rho(z,\gamma_p(x)) &= t_p \pm \epsilon \\ &= \sum_{k=0}^{p-1} s_k \pm \epsilon \\ &= \sum_{k=0}^{p-1} \rho(\gamma_{k+1}(x),\gamma_k(x)) \pm (2p+1)\epsilon.\end{aligned}$$

Thus

$$\begin{aligned}\psi_p(\underline{t},z) &= \sum_{\underline{\gamma}\in\Gamma^p(\underline{t})} \int_0^{2\pi} \prod_{j=1}^{p} 1_{\varphi_z^{-1}\gamma_j N}(\tanh t_j e^{i\theta})d\theta \\ &\leq \sum_{\underline{\gamma}\in\Gamma^p(\underline{t})} \int_0^{2\pi} 1_{\varphi_z^{-1}\gamma_p N}(\tanh t_j e^{i\theta})d\theta \\ &\leq \sum_{\underline{\gamma}\in\Gamma^p(\underline{t})} |\Lambda(\varphi_z^{-1}\gamma_p(x),\epsilon)| \quad \because\ \tanh t_j e^{i\theta} \in \varphi_z^{-1}\gamma_j N \ \Rightarrow\ \theta\in\Lambda(\varphi_z^{-1}\gamma_p(x),\epsilon), \\ &\ll \sum_{\underline{\gamma}\in\Gamma^p(\underline{t})} e^{-\rho(0,\varphi_z^{-1}\gamma_p x)} \\ &= \sum_{\underline{\gamma}\in\Gamma^p(\underline{t})} e^{-\rho(z,\gamma_p x)} \\ &\ll \sum_{\underline{\gamma}\in\Gamma^p(\underline{t})} \prod_{k=0}^{p-1} e^{-\rho(\gamma_k(x),\gamma_{k+1}(x))} \\ &\leq \prod_{k=0}^{p-1} \sum_{\gamma\in\Gamma,\ \rho(z,\gamma(x))=s_k\pm 2\epsilon} e^{-\rho(x,\gamma(x))} \\ &\ll \prod_{k=0}^{p-1} m(\Delta(x,4\epsilon)\cap\varphi_\Gamma^{-s_k}\Delta(x,4\epsilon)\end{aligned}$$

and

$$\begin{aligned}m(\bigcap_{j=0}^{p}\phi_\Gamma^{-t_j}\Delta(x,\epsilon)) &= \int_N \psi_p(\underline{t},z)dA(z) \\ &\ll \prod_{k=1}^{p} m(\Delta(x,4\epsilon)\cap\phi_\Gamma^{-s_k}\Delta(x,4\epsilon)). \qquad \boxtimes\end{aligned}$$

To complete the proof, we use a `word metric observation` from [19].

Word length.

Define the S-*word length* of $\gamma \in \Gamma$ by

$$\ell(\gamma) = \ell_S(\gamma) := \min\{N \geq 1 : \ \exists\ c_1, c_2, \ldots, c_N \in S,\ \gamma = c_1 c_2 \ldots c_N\}.$$

This gives rise to the *word metric* d_ℓ on Γ given by

$$d_\ell(\beta, \gamma) := \ell(\gamma b^{-1}).$$

As before, a set of form

$$C = [c_1, c_2, \ldots, c_n] := \{x \in \Omega : x_k = c_k\ \forall\ 1 \leq k \leq n\}$$

is called a *cylinder of length n*. Let

$$\mathcal{C}_n := \{\text{cylinders of length } n\}.$$

To each $C = [c_1, c_2, \ldots, c_n] \in \mathcal{C}_n$ corresponds $\gamma = \gamma_C := c_1 c_2 \ldots c_n \in \Gamma$ with $\ell(\gamma_C) = n$.

It is shown in [19] that $\exists\ M = M_\Gamma > 0$ so that

(♿) (i) $\rho(\gamma(0), \beta(0)) \ = \ M^{\pm} d_\ell(\gamma, \beta)$,

(ii) $\mu(C) = M^{\pm 1} e^{-\rho(0, \gamma(0))}$

where $\emptyset \neq C = [c_1, c_2, \ldots, c_n] \subset \Omega\ \&\ \gamma = \gamma_C := c_n c_{n-1} \ldots c_2 c_1$.

Thus,

$$\begin{aligned}\sum_{\gamma \in \Gamma_0,\ \ell(\gamma) = n} e^{-\rho(0, \gamma(0))} &= M^{\pm} \sum_{\gamma = c_1 c_2 \ldots c_n \in \Gamma_0,\ \ell(\gamma) = n} \mu([c_1, \ldots, c_n]) \ \ \text{by (♿)(ii)} \\ &= \sum_{C \in \mathcal{C}_n,\ \Theta(\gamma_C) = 0} \mu([c_1, \ldots, c_n]) \\ &= \mu([\phi_n = 0]) \\ &\asymp \frac{1}{n^{\frac{\kappa}{2}}} \quad \text{by } (\mathtt{LLT}).\end{aligned}$$

Fix $t,\ K > 0$. Suppose that $\mathfrak{g} \in \Gamma_0\ \&\ \rho(0, \mathfrak{g}(0)) = t \pm K$. Let $\ell(\mathfrak{g}) = N$. If $\gamma \in \Gamma,\ \rho(0, \gamma(0)) = t \pm K$, then $\rho(\mathfrak{g}(0), \gamma(0)) < 2K$, whence $d_\ell(\mathfrak{g}, \gamma) < 2MK$ and

$$\ell(\gamma) = N \pm 2MK.$$

Using this and (♿)(i)

$$\sum_{\gamma\in\Gamma_0,\ \rho(0,\gamma(0))=t\pm K} e^{-\rho(0,\gamma(0))} \le \sum_{n=N\pm 2MK}\ \sum_{C\in\mathcal{C}_n,\ \Theta(\gamma_C)=0} e^{-\rho(0,\gamma(0))}$$
$$\le M \sum_{n=N\pm 2MK}\ \sum_{C\in\mathcal{C}_n,\ \Theta(\gamma_C)=0} \mu([\phi_n=0])$$
$$\ll \frac{1}{N^{\frac{\kappa}{2}}} \asymp \frac{1}{t^{\frac{\kappa}{2}}}.$$

This is the upper estimation in (🚂) and proves admissibility of sets of form $\Delta(x,\epsilon)$. It follows that $\varphi_{V^{(2)}}$ is 1-nice and 2-dissipative.

It follows from (LLL) that the representing semiflow of $\varphi_{V^{(1)}}$ is 2-nice, whence also $\varphi_{V^{(1)}}$. By theorem 2.2, $\varphi_{V^{(1)}}$ is subsequence 2-rationally ergodic, whence 2-recurrent. ☑

Higher dimensional theorem 5.1. In the interest of simplicity, we stated and proved theorem 5.1 for surfaces. In fact the analogous statements hold for geodesic flows of hyperbolic manifolds of arbitrary dimension with constant negative curvature. The proof is the same. The geodesic flow of a compact manifold with constant negative curvature is an Anosov flow. The results of [9] [19], [21] and [22] apply and the analytic geometry lemmas have analogous multidimensional versions.

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